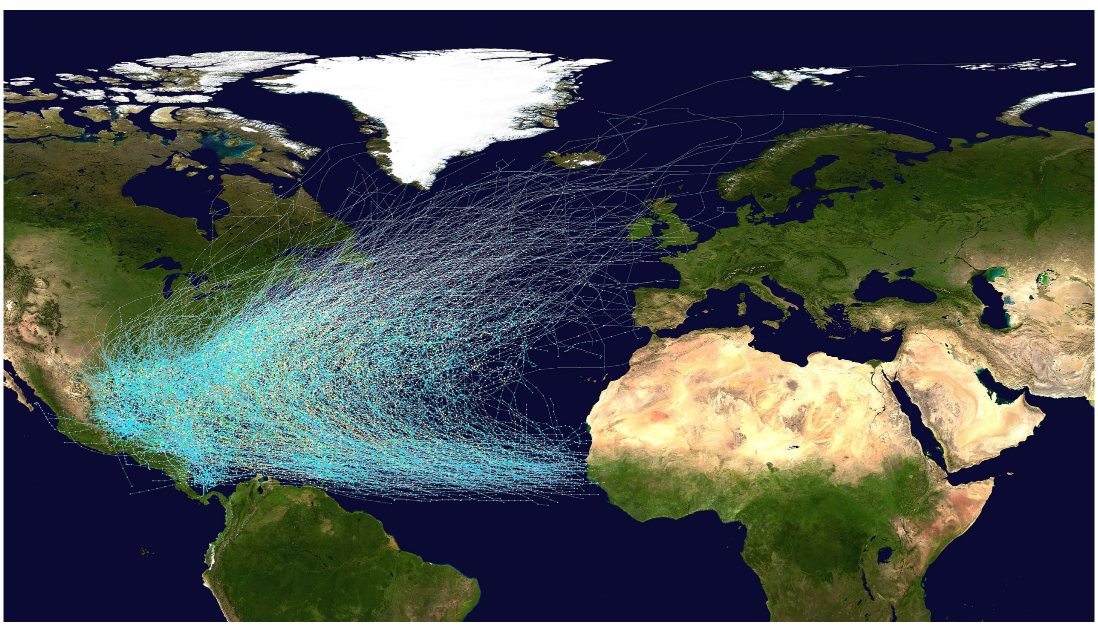
Grade 11-12 Math Circles
Nov. 2, 2022
The Mathematics of Climate Change 1

Introductory Comments

Climate change is the central challenge of our time, and in some ways the biggest challenge in the history of technological civilization. It is quite simply a problem that is far bigger than any one of us. Addressing climate change requires many branches of human activity. However, like many aspects of modern life, if you dig deep enough it's all math. This is hard to see in the case of climate change because the issue is so big, and to be perfectly honest, because it has been so politicized. In this set of lectures I will talk about little bits of the math of climate change, with the aim of having a peek behind the curtain, so that the next generation has a broader set of educational choices as they move from secondary to post-secondary education.



This image shows the hurricane tracks over the Atlantic over the past 170 years or so. You can see that even something as gigantic as a hurricane has a lot of irregularity in the path it travels along. This is a hallmark of weather and climate in general; the weather is chaotic (though with very clear patterns) and changes every 3-5 days (what is called a multi–day time scale), climate on the other hand typically changes slowly (decades, centuries, millenia...). However with the advent of human–caused (or anthropogenic) climate change climate has begun to change faster and some of the changes feed back on weather changes that impact people. So for example, a warmer Atlantic means a hurricane can miss Florida, head north to Canada and retain much more energy than 150 years ago, and hence something like hurricane Fiona could happen far more regularly going forward.

Stop and Think

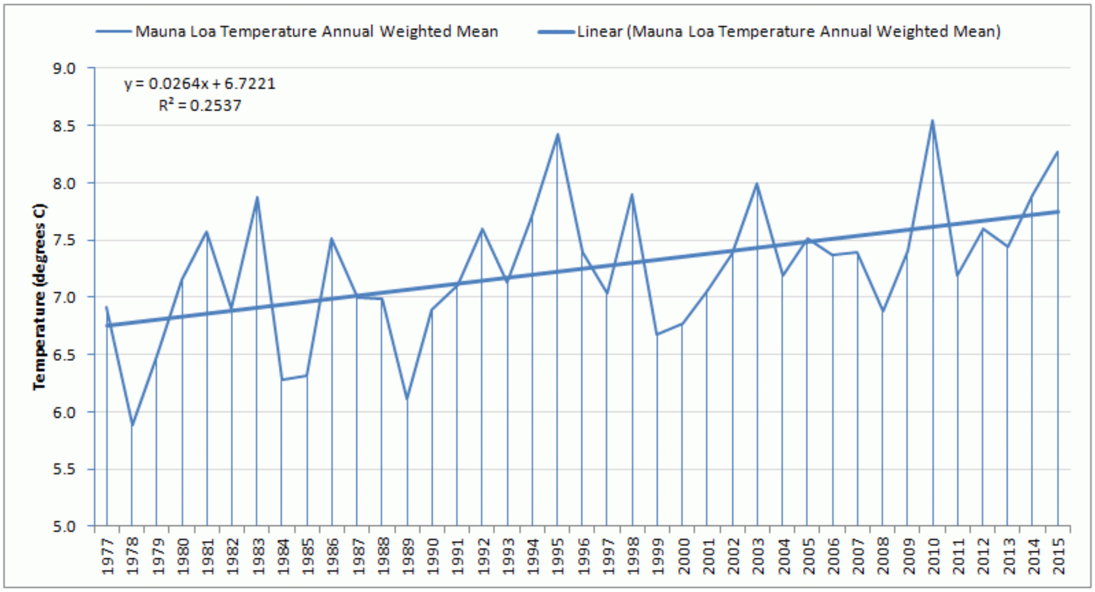
Think a little bit about air temperature. What are some of the time scales on which it changes? Try to write a few down along with estimates of how long they are. What is the shortest one you have written down? How about the longest one?

Thinking about Temperature Variations

Let's now think about the mathematics of describing temperature. If we measure at one fixed spot then temperature is a function of time. Let's write t for time, and let's follow the MKS system of units and measure time in seconds. The math-form of temperature is thus $T(t)$. This is a good spot to remind yourself of the mathematical definition of function. Because my undergraduate studies were in Math, I tend to immediately think of functions that are well behaved. For concreteness think of a periodic function like

$$T(t) = a\sin(\omega t)$$

where a is the amplitude of oscillation, and ω is the frequency of oscillation. This function is continuous for all real inputs t (you can think of this as "I can draw it without lifting my pencil"), and it is also differentiable (if you've seen calculus), and so a scientist would say "I can compute it's rate of change with time unambiguously". As far as difficulty, this seems like pretty easy stuff. But now think about a measurement of temperature, like the one shown in the figure below, it shows the annual average of temperature on top of Mauna Loa in Hawaii. I will tell you a little bit about why I am showing Mauna Loa in my presentation.



There are three important take aways from the figure. The most important is that real measurements are not the nice functions of mathematical theory, they are measured at discrete times, so you might add an index to our notation and write $T(t_i)$ where i is an natural number (positive integer). If you're less interested in the math, you might note two different scientific things. The first is that the data is a bit irregular; it hops around year to year. This is another manifestation of what we saw with the hurricane tracks. It is perhaps a bit surprising to you (it was to me) that even annually averaged temperatures vary this much, and speaks to just how complex the climate system is. The second thing to notice is that if you somehow get rid of that year to year variability there is a clear upward trend (the blue line has a positive slope).

Takeaways Section 2

The above illustrates the mathematical challenge we face: we need some functions to describe what is measured, and some ways to go back and forth between the mathematical world of continuous and differentiable functions and the real world of data. The Exercise below will let you explore some different facets of this.

Activity 1

Consider two functions $T_1(t) = 0.1t$ and $T_2(t) = 10t$. I will hand out a way to generate random numbers by flipping a coin. For times $t = 1, 2, 3, ...20$ generate artificial time series by using $T_1(t)$ and $T_2(t)$ but at each time add 1 if the coin comes up as "heads" and subtract 1 if the coin comes up as "tails". Graph your result, and describe how the graphs look different.

Discussion: (you can record what we talk about here):

Activity 2

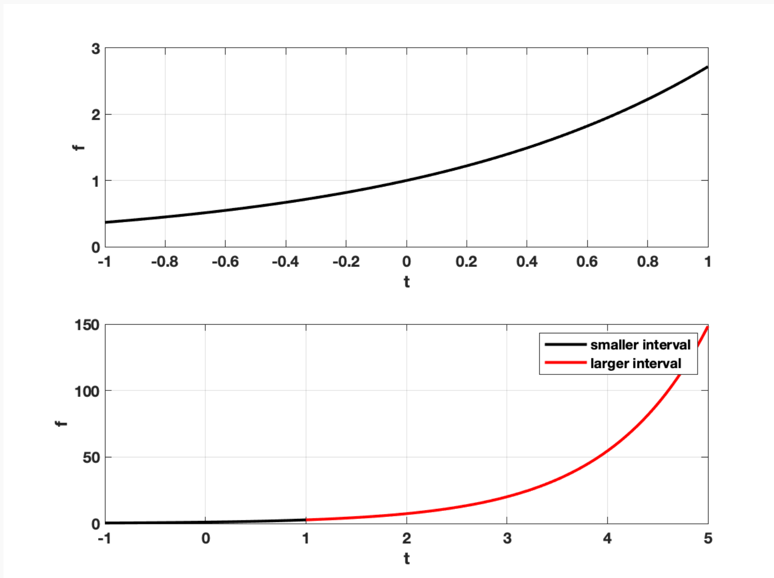
OK, now let's consider a different function. You've probably heard that climate change is an exponential process, and I talked a bit about it in my presentation. The exercises below remind you of some of the properties of exponentials, and you can start the activity by doing a few of those problems. Now let's return to the coin flipping way of introducing randomness and apply it to the function $T_3(t) = \exp(t/5) = e^{t/5}$. Start at $t = 0$ this time and do the coin flips for $t = 0, 1, 2, 3, ...20$. How do the results differ from the linear cases of Activity 1?

Discussion: (you can record what we talk about here):

Exercise 1

Consider the exponential function $f(t) = \exp(t) = e^t$. Sketch the function on the interval $-1 < t < 1$. Now repeat on the interval $-1 < t < 5$. Comment on how the two graphs differ. Focus on how the function in the smaller interval appears in the graph over the larger interval.

Exercise 1 Solution



Exercise 2

Consider the exponential function $f(t) = \exp(at) = e^{at}$. Discuss the effect of the changes of the parameter a on the graph of $f(t)$. In particular, estimate the time needed to increase f by a factor of 10.

Exercise 2 Solution

The attached figure shows the qualitative effect of changing a. To get the estimate of increasing by a factor of ten we first need a value to start from. Choosing $t = 0$ we get $f(0) = 1$ no matter what a is so that is quite convenient. Now we want to find t_1 so that $f(t_1) = 10$ so we need to use the natural logarithm:

$$exp(at_1) = 10 \implies at_1 = \ln(10) \implies t_1 = t_1 = \ln(10)/a.$$

If you want that in terms of numbers you get the estimate $t_1 \approx 2.3026/a$. The point is that the time to increase by a factor of ten is inversely proportional to a.